SUMS OF THREE CUBES

Trevor D. Wooley

1. Introduction. The set of integers represented as the sum of three cubes of natural numbers is widely expected to have positive density (see Hooley [7] for a discussion of this topic). Over the past six decades or so, the pursuit of an acceptable approximation to the latter statement has spawned much of the progress achieved in the theory of the Hardy-Littlewood method, so far as its application to Waring's problem for smaller exponents is concerned. Write $R(N)$ for the number of positive integers not exceeding N which are the sum of three cubes of natural numbers. Then by exploiting methods based on the use of diminishing ranges, Davenport [4] established that $R(N) \gg N^{13/15-\varepsilon}$, a bound which Davenport [5] himself subsequently improved to obtain $R(N) \gg N^{47/54-\varepsilon}$. It remained until the work of Vaughan for further improvement to be achieved. First, in work which may be considered as a natural development of Davenport's methods, Vaughan [11, 12] obtained the lower bound $R(N) \gg N^{19/21-\varepsilon}$. Later, as a consequence of his "new iterative method" involving the use of exponential sums over smooth numbers, Vaughan [13] obtained the sharper bound $R(N) \gg N^{11/12-\varepsilon}$ (see also Ringrose [10] for an intermediate estimate). Most recently, the author has developed an extension of the new iterative method in which fractional moments of exponential sums over smooth numbers are estimated non-trivially, and thereby (see Corollary B to Theorem 1.2 of Wooley [15]) has obtained the lower bound

$$R(N) \gg N^{1-\xi/3-\varepsilon},$$

where ξ denotes the positive root of the polynomial $\xi^3 + 16\xi^2 + 28\xi - 8$, so that $\xi = 0.24956813\dots$. The purpose of the present paper is to obtain a further modest sharpening in the lower bound for $R(N)$.

Theorem 1.1. *For each positive number ε, one has*

$$R(N) \gg N^{\alpha-\varepsilon},$$

where $\alpha = (166 - \sqrt{2833})/123$.

Research supported in part by NSF grant DMS-9622773 and a Fellowship from the David and Lucile Packard Foundation. This paper was written during the author's stay at the Institute for Advanced Study and Princeton University, supported by the 1998 Salem Prize.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

For comparison, one has the lower bound $\alpha > 0.916862$, whereas $1 - \xi/3 < 0.916811$. Although this improvement in the lower bound for $R(N)$ may be the smallest in history, it is to be hoped that the progress described herein may at least stimulate further progress in this stubborn problem. We remark that, subject to the truth of an unproved Riemann Hypothesis concerning certain Hasse-Weil L-functions, one has the conditional estimate $R(N) \gg N^{1-\varepsilon}$ due to Hooley [8, 9] and Heath-Brown [6]. Unfortunately, the latter L-functions are not yet known to possess an analytic continuation inside the critical strip, and thus the resolution of a Riemann Hypothesis seems a distant prospect.

We establish Theorem 1.1 in routine manner by exploiting a mean value estimate of independent interest. In order to discuss this estimate, we require some notation. Denote by $\mathcal{A}(P,R)$ the set of R-smooth numbers of size at most P, that is

$$\mathcal{A}(P,R) = \{n \in [1,P] \cap \mathbb{Z} \ : \ p|n \text{ and } p \text{ prime} \Rightarrow p \leq R\}. \tag{1.1}$$

As usual, we write $e(z)$ for $e^{2\pi i z}$, and define the smooth Weyl sum $f(\alpha) = f(\alpha; P, R)$ by

$$f(\alpha; P, R) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^3), \tag{1.2}$$

and the classical Weyl sum $F(\alpha) = F(\alpha; P)$ by

$$F(\alpha; P) = \sum_{1 \leq x \leq P} e(\alpha x^3). \tag{1.3}$$

In §2 we establish the estimates contained in the following theorem.

Theorem 1.2. *For each $\varepsilon > 0$, there exists a positive number $\eta = \eta(\varepsilon)$ such that whenever $R \leq P^\eta$, one has*

$$\int_0^1 \left| F(\alpha; P)^2 f(\alpha; P, R)^4 \right| d\alpha \ll P^{3+\delta_6+\varepsilon} \tag{1.4}$$

and

$$\int_0^1 |f(\alpha; P, R)|^5 d\alpha \ll P^{\frac{5}{2}+\delta_5+\varepsilon}, \tag{1.5}$$

where

$$\delta_6 = \frac{\sqrt{2833} - 43}{41} \quad \text{and} \quad \delta_5 = \frac{\sqrt{2833} - 49}{48}. \tag{1.6}$$

For comparison, Theorem 1.2 and Lemma 5.1 of Wooley [15] establish similar estimates to those of Theorem 1.2 with $\delta_6 = \xi$ and $\delta_5 = 3\xi/(8 + 2\xi)$, where ξ is the number defined in the opening paragraph. Earlier work of Vaughan [13, Theorem 4.4] had established the upper bound

$$\int_0^1 |f(\alpha; P, R)|^6 d\alpha \ll P^{13/4+\varepsilon},$$

and this yields the estimate (1.5) with $\delta_5 = 1/8$ via Schwarz's inequality. It may be useful to record that the values δ_5 and δ_6 recorded in (1.6) satisfy

$$\delta_5 = 0.08804028\dots \quad \text{and} \quad \delta_6 = 0.24941301\dots.$$

As will be familiar to experts, the upper bound (1.4) of Theorem 1.2 has immediate consequences for estimates concerning the exceptional set for sums of four cubes. Let $E(X)$ denote the number of natural numbers not exceeding X which are not the sum of four cubes of natual numbers. Then by following the argument of Brüdern [2], one readily establishes the estimate contained in the following theorem. We provide no further discussion of the proof of this theorem.

Theorem 1.3. *For each positive number ε, one has*

$$E(X) \ll X^{1-\beta+\varepsilon},$$

where $\beta = (422 - 6\sqrt{2833})/861$.

The aforementioned work of Brüdern [2] yields a similar conclusion with $\beta = 5/42$, this having been improved in Corollary B to Theorem 1.2 of Wooley [15] to $\beta = (4-6\xi)/21 < 0.119172$. For comparison, the value of β recorded in Theorem 1.3 satisfies $\beta > 0.119215$.

We establish the mean value estimates of Theorem 1.2 by means of the iterative method described in Wooley [15]. The key feature of the latter method is that it estimates non-trivially the fractional moments of smooth Weyl sums, and in the proof of Theorem 1.2 it is the fifth moment which plays the leading role. For the most part we follow the treatment applied in §5 of Wooley [15], but now we exploit sharper major arc estimates following the differencing operation in order to permit greater use to be made of the fifth moment. The sharper estimates presented in Theorem 1.2 lead to small improvements in all small moments of cubic smooth Weyl sums, and this topic we briefly discuss at the end of §2.

We use ε and η to denote sufficiently small positive numbers, and P to denote a large positive number depending at most on ε and η. The implicit constants in Vinogradov's well-known notation, $\ll$ and $\gg$, will depend at most on ε and η. We adopt the following convention concerning the numbers ε and R. Whenever ε or R appear in a statement, either implicitly or explicitly, we assert that for each $\varepsilon > 0$, there exists a positive number $\eta(\varepsilon)$ such that the statement holds whenever $R \le P^\eta$. Note that the "value" of ε, and η, may change from statement to statement, and hence also the dependency of implicit constants on ε and η. We observe that since our iterative methods will involve only a finite number of statements (depending at most on ε), there is no danger of losing control of implicit constants through the successive changes in our arguments.

2. The proof of Theorem 1.2. Before establishing the mean value estimates contained in Theorem 1.2, we must recall some notation from Wooley [15]. When s is a positive real number, define the mean value $U_s(P,R)$ by

$$U_s(P,R) = \int_0^1 |f(\alpha;P,R)|^s \, d\alpha.$$

We say that an exponent μ_s is *permissible* whenever the exponent has the property that, with the notational conventions defined above, one has $U_s(P,R) \ll P^{\mu_s+\varepsilon}$. It follows easily as in [15] that for each s, a permissible exponent μ_s exists satisfying $s/2 \le \mu_s \le s$. It is convenient to refer to an exponent δ_s as an *associated exponent* when $\mu_s = s/2 + \delta_s$ is permissible.

We provide associated exponents δ_5 by applying Lemma 5.1 of [15], which we record below in the following lemma.

Lemma 2.1. *Suppose that δ_6 is an associated exponent. Then the exponent $\delta_5 = 3\delta_6/(8+2\delta_6)$ is associated.*

It is in the analysis of the associated exponents δ_6 that our treatment differs from that of Wooley [15].

Lemma 2.2. *Suppose that δ_5 and δ_6 are associated exponents. Then the exponent δ_6' is associated, where*

$$\delta_6' = 2\max\left\{\frac{3+8\delta_5}{29+8\delta_5}, \frac{\delta_6}{4+\delta_6}\right\}. \tag{2.1}$$

Moreover, one has

$$\int_0^1 \left|F(\alpha;P)^2 f(\alpha;P,R)^4\right| d\alpha \ll P^{3+\delta_6'+\varepsilon}. \tag{2.2}$$

Proof. Initially, we follow the treatment of Lemma 5.2 of Wooley [15]. Let ϕ be a real number with $0 \le \phi \le 1/7$, and write

$$M = P^\phi, \quad H = PM^{-3} \quad \text{and} \quad Q = PM^{-1}.$$

Next define the exponential sum

$$F_1(\alpha) = \sum_{1\le z\le 2P}\sum_{1\le h\le H}\sum_{M<m\le MR} e(2\alpha h(3z^2+h^2m^6)),$$

and define the mean value $I(\mathcal{B})$, when $\mathcal{B}\subseteq[0,1)$, by

$$I(\mathcal{B}) = \int_{\mathcal{B}} \left|F_1(\alpha) f(\alpha;2Q,R)^4\right| d\alpha. \tag{2.3}$$

Then the inequality (5.3) of [15] yields the estimate

$$\int_0^1 \left|F(\alpha;P)^2 f(\alpha;P,R)^4\right| d\alpha \ll P^\varepsilon M^3\left(PMQ^2 + I([0,1))\right). \tag{2.4}$$

On considering the underlying diophantine equation, the integral on the left hand side of (2.4) provides an upper bound for $U_6(P,R)$, and hence the estimate (2.2) establishes that the exponent δ_6' defined in (2.1) is associated.

Let $\mathfrak{m}$ denote the set of points α in $[0,1)$ with the property that whenever there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$ and $|q\alpha - a| \le PQ^{-3}$, then one has $q > P$. Further, let $\mathfrak{M} = [0,1) \setminus \mathfrak{m}$. We aim to apply the Hardy-Littlewood method to estimate the mean value $I([0,1))$, and from this the desired upper bound (2.2) will follow.

We begin by estimating the contribution of the minor arcs $\mathfrak{m}$ to $I([0,1))$. By applying Hölder's inequality to (2.3), we obtain

$$I(\mathfrak{m}) \ll J^{1/5} U_5^{4/5}, \tag{2.5}$$

where

$$J = \int_{\mathfrak{m}} |F_1(\alpha)|^5 d\alpha \quad \text{and} \quad U_5 = \int_0^1 |f(\alpha; 2Q, R)|^5 d\alpha. \tag{2.6}$$

But by inequality (5.4) of [15] together with the argument of the proof of Lemma 3.7 of Vaughan [13], one has

$$J \le \Big(\sup_{\alpha \in \mathfrak{m}} |F_1(\alpha)| \Big)^3 \int_0^1 |F_1(\alpha)|^2 d\alpha \ll P^{\varepsilon} \Big((PM)^{1/2} H \Big)^3 (PMH).$$

Also, on recalling that δ_5 is an associated exponent, we have

$$U_5 \ll Q^{\frac{5}{2}+\delta_5+\varepsilon}.$$

Thus it follows from (2.5) that

$$I(\mathfrak{m}) \ll P^{\varepsilon} (PM)^{\frac{1}{2}} H^{\frac{4}{5}} Q^{2+\frac{4}{5}\delta_5}. \tag{2.7}$$

In order to provide a satisfactory estimate for $I(\mathfrak{M})$, we investigate an auxiliary mean value. Observe that $\mathfrak{M}$ is the union over $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying $(a,q)=1$ and $0 \le a \le q \le P$, of the intervals

$$\mathfrak{M}(q,a) = \{ \alpha \in [0,1) \ : \ |q\alpha - a| \le PQ^{-3} \}.$$

Define the function $\Delta(\alpha)$ for $\alpha \in [0,1)$ by

$$\Delta(\alpha) = \begin{cases} (q + Q^3 |q\alpha - a|)^{-1}, & \text{when } \alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}, \\ 0, & \text{otherwise,} \end{cases}$$

and define the mean value

$$K = \int_{\mathfrak{M}} \Delta(\alpha) |f(\alpha; 2Q, R)|^2 d\alpha. \tag{2.8}$$

Plainly,

$$|f(\alpha; 2Q, R)|^2 = \sum_{l \in \mathbb{Z}} \psi(l) e(l\alpha),$$

where $\psi(l)$ denotes the number of solutions of the equation $z_1^3 - z_2^3 = l$ with $z_1, z_2 \in \mathcal{A}(2Q, R)$. One evidently has

$$\psi(0) \ll Q \quad \text{and} \quad \sum_{l\in\mathbb{Z}} \psi(l) = f(0; 2Q, R)^2 \ll Q^2.$$

Applying the latter estimates within Lemma 2 of Brüdern [1], we deduce that

$$\int_{\mathfrak{M}} \Delta(\alpha)|f(\alpha; 2Q, R)|^2 d\alpha \ll Q^{\varepsilon-3}(PQ + Q^2) \ll PQ^{\varepsilon-2}. \tag{2.9}$$

Next we note that by Lemmata 3.1 and 3.4 of Vaughan [13], when $\alpha \in \mathfrak{M}$ one has

$$F_1(\alpha) \ll P^{\varepsilon}(PHM\Delta(\alpha)^{2/3} + PHM^{1/2}\Delta(\alpha)^{1/2}).$$

Then by combining (2.3) with (2.8) via Hölder's inequality, we obtain

$$\begin{aligned} I(\mathfrak{M}) \ll & P^{1+\varepsilon} HMK^{2/3} \Big(\int_0^1 |f(\alpha; 2Q, R)|^8 d\alpha \Big)^{1/3} \\ & + P^{1+\varepsilon} HM^{1/2}K^{1/2} \Big(\int_0^1 |f(\alpha; 2Q, R)|^6 d\alpha \Big)^{1/2}. \end{aligned}$$

Consequently, on recalling Hua's Lemma (see Lemma 2.5 of Vaughan [14]), and making use of (2.9) and our hypothesis that δ_6 is an associated exponent, we deduce that

$$I(\mathfrak{M}) \ll P^{1+\varepsilon} HM(PQ^{-2})^{2/3}(Q^5)^{1/3} + P^{1+\varepsilon} HM^{1/2}(PQ^{-2})^{1/2}(Q^{3+\delta_6})^{1/2}.$$

On recalling (2.7) and (2.4), we thus obtain the bound

$$\begin{aligned} \int_0^1 |F(\alpha; P)^2 f(\alpha; P, R)^4| d\alpha &\ll P^{\varepsilon} M^3 (PMQ^2 + I(\mathfrak{M}) + I(\mathfrak{m})) \\ &\ll P^{\varepsilon} M^3 Q^2 (\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4), \end{aligned} \tag{2.10}$$

where

$$\Phi_1 = PM, \quad \Phi_2 = (PM)^{\frac{1}{2}} H^{\frac{4}{5}} Q^{\frac{4}{5}\delta_5},$$
$$\Phi_3 = P^{5/3} HMQ^{-5/3}, \quad \Phi_4 = P^{3/2} HM^{1/2} Q^{(\delta_6 - 3)/2}.$$

In view of the definitions of M, H and Q, however, one finds that $\Phi_3 = PM^{-1/3} \le \Phi_1$, that whenever

$$\phi \ge \frac{3 + 8\delta_5}{29 + 8\delta_5},$$

one has $\Phi_1 \ge \Phi_2$, and that whenever $\phi \ge \delta_6/(4 + \delta_6)$, one has $\Phi_1 \ge \Phi_4$. Thus, on setting

$$\phi = \max \left\{ \frac{3 + 8\delta_5}{29 + 8\delta_5}, \frac{\delta_6}{4 + \delta_6} \right\}, \tag{2.11}$$

we deduce from (2.10) that

$$\int_0^1 |F(\alpha;P)^2 f(\alpha;P,R)^4|d\alpha \ll P^{1+\varepsilon}M^4Q^2 = P^{3+2\phi+\varepsilon},$$

whence the desired estimate (2.2) follows immediately from (2.11).

Theorem 1.2 follows by applying Lemmata 2.1 and 2.2 iteratively, as we now demonstrate.

The proof of Theorem 1.2. Suppose that δ_t $(t=5,6)$ are associated exponents. Then by applying Lemmata 2.1 and 2.2 repeatedly, we obtain a sequence of such associated exponents, $\delta_t^{(r)}$ $(t=5,6)$, with the property that $\delta_t^{(0)}=\delta_t$ and for $r\geq 0$,

$$\delta_5^{(r+1)} = \frac{3\delta_6^{(r)}}{8+2\delta_6^{(r)}} \quad \text{and} \quad \delta_6^{(r+1)} = 2\max\left\{\frac{3+8\delta_5^{(r)}}{29+8\delta_5^{(r)}}, \frac{\delta_6^{(r)}}{4+\delta_6^{(r)}}\right\}. \tag{2.12}$$

If the second expression in the maximum defines $\delta_6^{(r+1)}$ in (2.12) for infinitely many values of r, then plainly $\delta_6^{(r)} \longrightarrow 0$ as $r\longrightarrow\infty$, and likewise for $\delta_5^{(r)}$. We may therefore suppose that for all sufficiently large r, it is the first expression which defines $\delta_6^{(r+1)}$ in (2.12). Then taking the limit as $r\longrightarrow\infty$, we deduce that the exponents δ_5^* and δ_6^* are associated, where δ_5^* and δ_6^* satisfy the equations

$$\delta_5^* = \frac{3\delta_6^*}{8+2\delta_6^*} \quad \text{and} \quad \delta_6^* = 2\frac{3+8\delta_5^*}{29+8\delta_5^*}.$$

It follows that δ_6^* is the smaller zero of the polynomial $41\xi^2+86\xi-24$, whence

$$\delta_6^* = \frac{\sqrt{2833}-43}{41} \quad \text{and} \quad \delta_5^* = \frac{\sqrt{2833}-49}{48}.$$

The estimates (1.4)-(1.6) now follow directly from Lemmata 2.1 and 2.2.

Theorem 1.1 follows immediately from Theorem 1.2 by means of an application of Cauchy's inequality. Since this argument is so often suppressed, we briefly describe the details for the benefit of inexperienced readers. We take $P=N^{1/3}$, and $R=N^\eta$ with $\eta=\eta(\varepsilon)$ a sufficiently small positive number. Then on writing $r(n)$ for the number of representations of the natural number n in the shape $n=x^3+y^3+z^3$, with $1\leq x\leq P$ and $y,z\in\mathcal{A}(P,R)$, one finds that

$$R(N) \geq \sum_{\substack{1\leq n\leq N\\ r(n)>0}} 1, \qquad \sum_{1\leq n\leq N} r(n) \gg P^3,$$

and

$$\sum_{1\leq n\leq N} r(n)^2 \leq \int_0^1 |F(\alpha;P)^2 f(\alpha;P,R)^4|d\alpha \ll P^{3+\delta_6+\varepsilon},$$

where δ_6 is the real number defined in (1.6). Thus, since by Cauchy's inequality,

$$\Big(\sum_{1\le n\le N} r(n)\Big)^2 \le \Big(\sum_{\substack{1\le n\le N\\ r(n)>0}} 1\Big)\Big(\sum_{1\le n\le N} r(n)^2\Big),$$

we deduce that

$$R(N) \gg (P^{3+\delta_6+\varepsilon})^{-1}(P^3)^2 = N^{1-(\delta_6+\varepsilon)/3}.$$

The conclusion of Theorem 1.1 is thus an immediate corollary of the estimate (1.4) of Theorem 1.2.

We conclude with a brief discussion of permissible exponents μ_s for $4 < s < 8$. This topic is investigated in detail in §4 of Brüdern and Wooley [3]. One finds, in particular, that methods currently available to us yield permissible exponents μ_s for $5 < s < 6$ which simply interpolate linearly between μ_5 and μ_6, and indeed a similar situation occurs for $6 < s < 6.5$. The explanation for this phenomenon is clear. One may provide bounds for μ_6 by means of the mean value estimate (2.4), and thus the exponential sum $F_1(\alpha)$, involving variables running over complete intervals, plays a prominent role. When s is not an even integer, the relevant mean values involve an analogue of $F_1(\alpha)$ in which certain linear combinations of variables are restricted to be smooth, and thus minor arc bounds for this analogue of $F_1(\alpha)$ are too weak to be of use. It thus transpires that the method of estimating μ_s when $s = 6$ is so much more efficient than that available for neighbouring values of s, that convexity arguments triumph close to $s = 6$. This phenomenon dictates that when estimating the mean value $I(\mathfrak{m})$ defined by (2.3), applications of Hölder's inequality which exploit mean values $U_t(P, R)$ should yield exponents μ_6 which are local extrema when $t = 5$, 6 and 6.5. The only obstacles to such a conclusion arise when estimating the contribution of the major arcs in the Hardy-Littlewood dissection, but in the present situation such obstacles have been removed. Thus we believe that the conclusion of Theorem 1.2 is the best available within the compass of our methods.

As noted in the introduction, the estimates of Theorem 1.2 may be recycled within §4 of Brüdern and Wooley [3]. However, the new permissible exponents obtained for $4 < s \le 7.365$ improve on those of §4 of [3] only in the 4th, 5th or 6th decimal places, and thus we avoid further discussion of this matter herein.

References

1. J. Brüdern, *A problem in additive number theory*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 27–33.
2. J. Brüdern, *On Waring's problem for cubes*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 229–256.
3. J. Brüdern and T. D. Wooley, *On Waring's problem for cubes and smooth Weyl sums* (submitted).
4. H. Davenport, *On Waring's problem for cubes*, Acta Math. **71** (1939), 123–143.
5. H. Davenport, *Sums of three positive cubes*, J. London Math. Soc. **25** (1950), 339–343.
6. D. R. Heath-Brown, *The circle method and diagonal cubic forms*, Philos. Trans. Roy. Soc. London Ser. A **356** (1998), 673–699.

7. C. Hooley, *On some topics connected with Waring's problem*, J. Reine Angew. Math. **369** (1986), 110–153.
8. C. Hooley, *On Waring's problem*, Acta Math. **157** (1986), 49–97.
9. C. Hooley, *On Hypothesis K^* in Waring's problem*, Sieve methods, exponential sums and their applications in number theory (Cardiff, 1995), London Math. Soc. Lecture Note Ser. 237, Cambridge University Press, Cambridge 1997, pp. 175–185.
10. C. J. Ringrose, *Sums of three cubes*, J. London Math. Soc. (2) **33** (1986), 407–413.
11. R. C. Vaughan, *Sums of three cubes*, Bull. London Math. Soc. **17** (1985), 17–20.
12. R. C. Vaughan, *On Waring's problem for cubes*, J. Reine Angew. Math. **365** (1986), 122–170.
13. R. C. Vaughan, *A new iterative method in Waring's problem*, Acta Math. **162** (1989), 1–71.
14. R. C. Vaughan, *The Hardy-Littlewood method, 2nd edition*, Cambridge University Press, Cambridge, 1997.
15. T. D. Wooley, *Breaking classical convexity in Waring's problem: Sums of cubes and quasi-diagonal behaviour*, Inventiones Math. **122** (1995), 421–451.

Department of Mathematics, University of Michigan, East Hall, 525 East University Avenue, Ann Arbor, Michigan 48109-1109, U.S.A.

E-mail address: wooley@math.lsa.umich.edu